Design and Evaluation of Sensor Systems for State and Parameter Estimation

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Sensors are important parts of feedback control designs and system identification functions. Practical sensors are, however, subject to a variety of errors that degrade state and parameter estimation accuracies. An analytical technique is developed to determine the effect of sensor errors on estimation accuracies. The analytical technique does not require Monte Carlo simulations and enables evaluation of the effects of several errors efficiently. The power of the suggested method results from propagation of the information matrix (inverse of the covariance matrix) rather than the covariance matrix, as is often done in most work on state estimation and parameter identification. Simpler algorithms result because of the additive property of the information matrix. Examples indicate the applicability of the techniques presented.

Introduction

SENSOR specification is becoming an increasingly more important aspect of subsystem integration. The basic problem is to select those instruments and associated accuracies that provide data for control, state estimation, and parameter estimation functions, required to improve overall system performance. A fundamental tradeoff exists, however, between the accuracy of a sensor and its cost. Accuracy itself is characterized by specific types of errors which cause the sensor output to deviate from the sensed variable (Table 1). Minimization of these types of errors produces the increase in cost of a particular sensor.

To some extent, there is a tendency to specify the best available sensor regardless of the cost. The result is an unmatched set of sensors and channels where most of the errors are produced by only a few channels or sensors, and the remaining elements are too accurate for the requirements. These latter elements could be degraded substantially with associated cost savings and without significant deterioration in overall mission performance. On the other hand, system capabilities can be advanced significantly by improving only a small number of subsystems. Also, effective fault tolerance may be obtained by using a redundant hardware for only the most critical systems.

The determination of subsystems most responsible for overall estimation error requires the computation of the sensitivity of parameter/state estimation error to each component of sensor/channel error. A direct approach is the Monte Carlo simulation method. In this method, the estimation procedure is simulated, together with the system dynamics. Errors are introduced one at a time, and the states and parameter estimation errors are computed for a number of runs. Ensemble average of these runs give the effect of one source of sensor/channel error on state/parameter estimation error. Since it may be necessary to make at least 100 runs for each error source, the procedure is very expensive for all but the simplest sensor systems. The advantage of the method is that it gives a sum of all of the errors, including numerical inaccuracies, etc. This method has been developed and applied extensively by Sorensen et al.[1] and others.

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To evaluate the sensitivities of state/parameter estimator to a large number of common sensor/channel errors efficiently, analytical techniques are necessary. Some work has been done in this area. Bierman[2] developed an analytical procedure for determining state estimation errors due to biases in state equations. An expression was derived for state estimation errors if the biases are ignored and if the biases do not exist. Bierman's approach implicitly made the assumption that the biases have a priori Gaussian distributions with known covariances. This can be a serious assumption because biases tend to be jump processes, as indicated by the works of Willsy and Jones.[3] Asher et al.[4] showed analytic expression for state estimation errors in reduced-order filters. Both of these approaches apply only to linear systems. Gupta[5] presented a procedure for computing bias and mean-square errors in general estimators. In that paper, a maximum-likelihood method was used instead of Bayes' approach, used by Bierman. The advantage of the maximum-likelihood method is that no assumption needs to be made about the distribution of error parameters. This is extremely important in nonlinear systems, where higher order expansions may be required to get the desired level of accuracy. Note that this does not invalidate Bierman's approach. In fact, in simple cases, our results will be equivalent to those obtained by Bierman. This paper extends previous results to evaluate the effects of sensor errors on state/parameter estimation errors in linear as well as nonlinear systems.

Problem Discussion

We consider a system (e.g., a vehicle) whose $n \times 1$ state vector x follows the dynamic equations

$$\dot{x}=f(x,u,\theta,\phi)+w(t); \qquad 0\leq t\leq T \tag{1}$$

Table 1 Typical errors in accelerometers and gyro outputs

Accelerometer (integrating)	
K_x	= scale factor
K_{xz}	= cross-axis sensitivity
E_w	= random uncertainty
B	= bias
$\delta_x,\delta_y,\delta_z$	= error in location relative to c.g.
Gyro (rate integrating)	
K_θ	= scale factor
K_{qx}	= anisotropy
K_{qr}	= anisoinertia
R	= drift bias
E_θ	= random uncertainty
(Torque rectification, threshold)	

where u is a $q\times 1$ vector of inputs to the system, θ is a $m\times 1$ vector of parameters of interest (e.g., aerodynamic parameters of an aircraft), ϕ is a $m'\times 1$ vector of error parameters (e.g., scale factor and nonlinearities in angle-of-attack vane), and $w(t)$ is a $n\times 1$ vector of noise inputs. Error parameters ϕ result from modeling errors, deterministic disturbances, or feedback of noisy signals in control channels. For our discussion, the noise will be assumed zero mean white Gaussian (ZMWG) with intensity Q. In general, sensors measure certain functions of states and inputs. Their outputs also may depend on parameters of interest and error parameters and usually are contaminated by random noise:

$$y(t)=h(x,u,\theta,\phi)+v(t); \qquad 0\leq t\leq T \tag{2}$$

$y(t)$, the sensor output, is a $p\times 1$ vector, and $v(t)$ is a zero mean white Gaussian noise process with intensity R. We assume continuous measurements, although discrete measurements are a straightforward extension.

Error parameters ϕ represent two major effects: 1) the modeling errors in state equations, and 2) parts of instrument errors which remain relatively constant over short intervals of time. These types of instrument errors are referred to as systematic errors, as opposed to random errors represented by ZMWG noise process $v(t)$. Typically, basis, scale factor, drift, etc., fall in this category.

Systematic errors and random errors have different effects on estimation accuracies. Therefore, in the context of state estimation and parameter identification, it is important to distinguish between random instrument errors and systematic instrument errors. Random errors produce a scatter in parameter and state estimates. In parameter identification, this scatter may be reduced by increasing the data length. Systematic errors can be handled in a number of ways. If they are ignored (set to a priori values), biased state and parameter estimates result. The bias can be reduced substantially by an accurate calibration of the instruments immediately prior to the test and recalibration at periodic intervals for long tests. Alternatively, since systematic errors remain essentially constant during the test, they may be estimated simultaneously with states and/or parameters of interest. Although the estimates are unbiased, they have a larger scatter because the same information is divided over a larger number of unknowns. In addition, the real-time computation requirements also may increase significantly. This indicates basic tradeoffs among improved hardware requirements, additional software/computation requirements, and estimation accuracy. The next two sections develop an analytical technique to evaluate the effects of ϕ and $v(t)$ on state/parameter estimation accuracies and to provide an optimal tradeoff between hardware accuracy (and cost) and software complexity.

Errors in State Estimates

In state estimation problems, the state and measurement equations do not contain unknown parameters θ; therefore, these equations simplify to

$$\dot{x}=f(x,u,\phi)+w(t) \tag{3a}$$

$$y=h(x,u,\phi)+v(t) \tag{3b}$$

A Kalman filter is an efficient method for state estimation, when system and sensor models and error covariances are known exactly. Since this is also a widely used technique, our error analysis will be based on this implementation of the state estimator. It should be pointed out that our error analysis approach is quite general and is applicable readily to any estimation algorithm, optimal or suboptimal (see also Ref. 5). The Kalman filter formulation is selected with a clear understanding of its limitations to demonstrate the power and generality of information matrix propagations.

When systematic errors are present, two approaches are applicable. In the first approach, the error parameters are fixed at a priori calibration values, while in the second, the error parameters are estimated concurrently with the state variables. We now show the effects of each approach on state estimation errors.

State Estimation Errors When Systematic Error Terms Are Set to Calibration Values

Let ϕ_c be the calibration values of systematic errors. The extended Kalman filter for state estimation would, then, be based on Eq. (3), with ϕ set to ϕ_c. The Kalman state estimator follows the equations

$$\dot{\hat{x}}=f(\hat{x},u,\phi_c)+K(\phi_c)[y-h(\hat{x},u,\phi_c)] \tag{4a}$$

$$\hat{x}(0)=\bar{x}(0) \tag{4b}$$

where the Kalman filter gain $K(\phi_c)$ is obtained from a solution of

$$\dot{M}(\phi_c)=-M(\phi_c)F(\phi_c)-F^T(\phi_c)M(\phi_c)$$
$$+H^T(\phi_c)R^{-1}H(\phi_c)-M(\phi_c)QM(\phi_c) \tag{5a}$$

$$M(\phi_c)\,|_{t=0}=P^{-1}(0) \tag{5b}$$

$$K(\phi_c)=M^{-1}(\phi_c)H^TR^{-1} \tag{6}$$

where $P(0)$ is the initial covariance of state estimate $x(0)$, M is the Fisher information matrix, and

$$F(\phi_c)=\frac{\partial f(\hat{x},u,\phi)}{\partial\hat{x}}\Big|_{\phi=\phi_c} \tag{7a}$$

$$H(\phi_c)=\frac{\partial h(\hat{x},u,\phi)}{\partial\hat{x}}\Big|_{\phi=\phi_c} \tag{7b}$$

Since the real system follows Eq. (3), the state estimation error is given by

$$\dot{\tilde{x}}=f(x,u,\phi)-f(\hat{x},u,\phi_c)-K(\phi_c)\{h(x,u,\phi)$$
$$-h(\hat{x},u,\phi_c)+v(t)\}+w(t) \tag{8}$$

where

$$\tilde{x}=x-\hat{x} \tag{9}$$

Expanding each term in a Taylor series about $(\hat{x},\phi_c)$ and retaining only the first-order term, we get

$$\dot{\tilde{x}}\approx F(\phi_c)\tilde{x}+\frac{\partial f}{\partial\phi}(\phi-\phi_c)-K(\phi_c)H(\phi_c)\tilde{x}$$
$$-K(\phi_c)\frac{\partial h}{\partial\phi}(\phi-\phi_c)-K(\phi_c)v(t)+w(t) \tag{10}$$

We divide $\tilde{x}$ into two parts:

$$\tilde{x}=\tilde{x}_r+\tilde{x}_b \tag{11}$$

where $\tilde{x}_r$ and $\tilde{x}_b$ are random and bias components of state estimation error. Clearly,

$$\dot{\tilde{x}}_r=[F(\phi_c)-K(\phi_c)H(\phi_c)]\tilde{x}_r-K(\phi_c)v(t)+w(t) \tag{12a}$$

$$\dot{\tilde{x}}_b=[F(\phi_c)-K(\phi_c)H(\phi_c)]\tilde{x}_b+\left[\frac{\partial f}{\partial\phi}-K(\phi_c)\frac{\partial h}{\partial\phi}\right](\phi-\phi_c) \tag{12b}$$

Covariance of the random component $\tilde{x}_r$ of the total estimation error is M^{-1}, where M is the solution of the Riccati equation, Eq. (5). The bias error may be obtained by solving Eq. (12). Equation (12b) does not contain any random noise term explicitly and therefore may be solved directly to give a time history of $\tilde{x}_b$. To determine the sensitivities of state errors to systematic errors, we introduce an $n \times m$ matrix X_b, which is a solution of the following equation:

$$\dot{X}_b = [F(\phi_c) - K(\phi_c)H(\phi_c)]X_b + \left[\frac{\partial f}{\partial \phi} - K(\phi_c)\frac{\partial h}{\partial \phi}\right] \quad (13)$$

Then, since Eq. (13) is linear,

$$\tilde{x}_b = X_b(\phi - \phi_c) \quad (14)$$

Since the random errors are uncorrelated with systematic errors, mean-square estimation error of x is given by

$$\text{mse}(\tilde{x}) = \underbrace{M^{-1}(\phi_c)}_{\substack{\text{state estimation}\\\text{when there are}\\\text{no systematic errors}}} + \underbrace{X_b(\phi-\phi_c)(\phi-\phi_c)^T X_b^T}_{\substack{\text{additional state estimation}\\\text{errors when the systematic}\\\text{errors are set to a priori}\\\text{calibration values}}} \quad (15)$$

It is important to point out, at this point, the errors in a state estimator not based on a Kalman filter. Equations (4) and (8-12) do not depend on the method for selecting $K(\phi_c)$. Clearly, the bias in the estimator is given by Eqs. (13) and (14), whether the gain $K(\phi_c)$ is based on Eq. (5) or not. The random error in state, $\tilde{x}_r$, still follows Eq. (12a). However, since $\tilde{x}_r$ and $\hat{x}$ may then be correlated even for linear systems, the covariance of the random component of error is not necessarily M^{-1}. It can be computed in a straightforward manner.[4] Therefore, only the first term in Eq. (15) is affected when a Kalman gain is not used.

A more accurate estimate of the state estimation error is obtained by retaining higher order terms in Eq. (10). For scalar state equations, retention of second-order terms gives

$$\begin{aligned}\dot{\tilde{x}} = {} & [F(\phi_c) - K(\phi_c)H(\phi_c)]\tilde{x} + \left[\frac{\partial f}{\partial \phi} - K(\phi_c)\frac{\partial h}{\partial \phi}\right](\phi - \phi_c) \\ & - K(\phi_c)v(t) + w(t) + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2} - K(\phi_c)\frac{\partial^2 h}{\partial x^2}\right]\tilde{x}^2 \\ & + \left[\frac{\partial^2 f}{\partial x \partial \phi} - K(\phi_c)\frac{\partial^2 h}{\partial x \partial \phi}\right](\phi - \phi_c)\tilde{x} \\ & + (\phi - \phi_c)^T\left[\frac{\partial^2 f}{\partial \phi^2} - K(\phi_c)\frac{\partial^2 h}{\partial \phi^2}\right](\phi - \phi_c) \end{aligned} \quad (16)$$

This is not a linear equation in $\tilde{x}$ and is therefore difficult to solve. A good approximation in many engineering problems is to replace $\tilde{x}$ in the last three terms of Eq. (16) by using Eqs. (11) and (14). This makes those terms functions of quadratic errors in ϕ, and their effects can be evaluated directly, because the approximated equation is linear in $\tilde{x}$. Although higher order expansions may be used, in engineering problems it is rarely necessary.

State Estimation Errors When Systematic Errors Are Estimated Concurrently with State Variables

When the extended Kalman filter is used to estimate instrument errors as well as state variables, the system state vector must be augmented to include error parameters. In addition to state equations, Eqs. (3), we must also use $\dot{\phi} = 0$. It is assumed that the initial value of ϕ is ϕ_c, and its estimation error covariance is large. The extended Kalman filter for x and ϕ is

$$\dot{\hat{x}} = f(\hat{x}, u, \hat{\phi}) + K_1[y - h(\hat{x}, u, \hat{\phi})]; \qquad \hat{x}(0) = \bar{x}(0) \quad (17a)$$

$$\dot{\hat{\phi}} = K_2[y - h(\hat{x}, u, \hat{\phi})]; \qquad \hat{\phi}(0) = \phi_c \quad (17b)$$

The information matrix for state x and parameters ϕ is of size $(n+m') \times (n+m')$. It follows the equation

$$\begin{aligned}\dot{M} = {} & -M\left[\begin{array}{c|c} F(\hat{\phi}) & \frac{\partial f}{\partial \phi} \\ \hline 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} F(\hat{\phi}) & \frac{\partial f}{\partial \phi} \\ \hline 0 & 0 \end{array}\right]^T M \\ & + \left[\begin{array}{c|c} H(\hat{\phi}) & \frac{\partial h}{\partial \phi} \end{array}\right]^T R^{-1}\left[\begin{array}{c|c} H(\hat{\phi}) & \frac{\partial h}{\partial \phi} \end{array}\right] - M\left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right]M \end{aligned} \quad (18)$$

$$\left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right] = M^{-1}\left[\begin{array}{c|c} H(\hat{\phi}) & \frac{\partial h}{\partial \phi} \end{array}\right]^T R^{-1} \quad (19)$$

Since the covariance of the initial estimate of ϕ is large,

$$M(0) = \left[\begin{array}{c|c} P^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] \quad (20)$$

Writing

$$M \triangleq \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12} & M_{22} \end{array}\right]$$

we get

$$\dot{M}_{11} = -M_{11}F(\hat{\phi}) - F^T(\hat{\phi})M_{11} + H^T(\hat{\phi})R^{-1}H(\hat{\phi}) - M_{11}QM_{11} \quad (21)$$

$$\dot{M}_{12} = -M_{11}\frac{\partial f}{\partial \phi} - F^T(\hat{\phi})M_{12} + H^T(\hat{\phi})R^{-1}\frac{\partial h}{\partial \phi} - M_{11}QM_{12} \quad (22)$$

$$\dot{M}_{22} = -M_{12}^T\frac{\partial f}{\partial \phi} - \left(\frac{\partial f}{\partial \phi}\right)^T M_{12} + \left(\frac{\partial h}{\partial \phi}\right)^T R^{-1}\frac{\partial h}{\partial \phi} - M_{12}^T Q M_{12} \quad (23)$$

Note that M_{11} follows exactly the same equation as $M(\phi_c)$, except that estimated values of ϕ are used in the M_{11} equation instead of the calibration values. Also, the M_{11} equation is independent of M_{12} and M_{22} and therefore remains the same whether the systematic errors are estimated or not. M^{-1} is the mean-square estimation error in x and ϕ. The mean-square estimation error in x is therefore

$$\text{mse}(x) = \underbrace{M_{11}^{-1}}_{\substack{\text{state estimation errors}\\\text{when there are no}\\\text{systematic errors}}} + \underbrace{M_{11}^{-1}M_{12}(M_{22} - M_{12}^T M_{11}^{-1}M_{12})^{-1}M_{12}^T M_{11}^{-1}}_{\substack{\text{additional state estimation errors}\\\text{caused by simultaneous estimation}\\\text{of systematic errors}}} \quad (24)$$

It is easy to show that M_{22} always increases. In general, the contribution of the second term to the covariance of x decreases with increasing time.

Table 2 Effect of systematic instrument errors on state estimation errors

	Bias	Variance	Mean-square error
Systematic errors set to a priori calibration values	$X_b(\phi-\phi_c)$ $X_b=M_{11}^{-1}M_{12}$	M_{11}^{-1}	$M_{11}^{-1}+X_b(\phi-\phi_c)\times(\phi-\phi_c)^T X_b^T$
Systematic errors identified simultaneously with state	0	$M_{11}^{-1}+X_b(M_{22}-M_{12}^T M_{11}^{-1}M_{12})^{-1}X_b^T$ $X_b=M_{11}^{-1}M_{12}$	$M_{11}^{-1}+X_b(M_{22}-M_{12}^T M_{11}^{-1}M_{12})^{-1}X_b^T$

Table 2 summarizes the effect of systematic errors on state estimation accuracies. It is easy to verify that

$$X_b=M_{11}^{-1}M_{12} \tag{25}$$

except for the difference in the state and systematic error values where linearization is performed. The first term is essentially equal in the two cases. The second term should be compared to determine the useful approach in a specific estimation problem. The systematic errors should be identified if the actual values of instrument systematic errors are likely to be significantly different from the calibration values.

Parameter Estimation Errors

The sensitivity of parameter estimation errors to sensor inaccuracies may be divided into two parts: the effects of systematic sensor errors, and the effects of random errors. Systematic errors in instruments and nonrandom modeling errors may, as shown before, be treated in one of two ways: either setting them to calibration values, or estimating them simultaneously with parameters. The random errors must be treated differently.

Estimation errors will be computed assuming that a maximum-likelihood scheme is used for parameter estimation. Since the maximum likelihood is optimal in a variety of circumstances, this analysis may give the lower bound on the estimation error. Other estimators may be treated in a similar manner using techniques developed by Gupta.[5] Again, it should be mentioned that the proposed method is general and may be applied to other parameter identification methods with suitable modification.

The likelihood technique is based on the concept of the negative log-likelihood function (NLLF), which is a function of the parameters to be identified θ and error parameters ϕ. The NLLF, $J(\theta,\phi)$, is given by

$$J(\theta,\phi)=\int_0^T \{\nu^T(t)R^{-1}\nu(t)+\log|R|\}\mathrm{d}t \tag{26a}$$

$$\nu(t)=y(t)-h[\hat{x}(t),u(t),\theta,\phi] \tag{26b}$$

where $\hat{x}(t)$ is obtained from a Kalman filter, shown in Eq. (4). Values of θ and ϕ which minimize $J(\theta,\phi)$ are the maximum-likelihood parameter estimates.

Parameter Estimation Errors with Systematic Errors Set to Calibration Values

When the error parameters are set to a priori values ϕ_c, the estimates of parameters θ are obtained by minimizing $J(\theta,\phi_c)$ with respect to θ; i.e., θ is a solution of

$$\frac{\partial J}{\partial \theta}(\theta,\phi_c)=0 \tag{27}$$

Expanding Eq. (27) about (θ, ϕ) we get

$$\frac{\partial J}{\partial \theta}(\hat{\theta},\phi)+\frac{\partial^2 J}{\partial\theta\partial\phi}(\theta,\phi)(\phi_c-\phi)+\frac{\partial^2 J}{\partial\theta^2}(\theta,\phi)(\hat{\theta}-\theta) + \text{higher-order terms}=0 \tag{28}$$

It easily is shown that the estimate $\hat{\theta}$ is biased (see Ref. 6). The bias, variance, and mean-square error of this estimator are [neglecting the higher-order terms in Eq. (28)]

$$E(\hat{\theta}-\theta)=M_{11}^{-1}M_{12}(\phi-\phi_c) \tag{29a}$$

$$\mathrm{var}(\hat{\theta}-\theta)=M_{11}^{-1} \tag{29b}$$

$$\mathrm{mse}(\hat{\theta}-\theta)=\underbrace{M_{11}^{-1}}_{\text{estimation error even when systematic errors are absent}}+\underbrace{M_{11}^{-1}M_{12}(\phi-\phi_c)(\phi-\phi_c)^T M_{12}^T M_{11}^{-1}}_{\text{additional errors when systematic errors are set to a priori calibration values}} \tag{29c}$$

where

$$M_{11}=\frac{\partial^2 J}{\partial\theta^2},\qquad M_{12}=\frac{\partial^2 J}{\partial\theta\partial\phi},\qquad M_{22}=\frac{\partial^2 J}{\partial\phi^2} \tag{30}$$

A more accurate estimate of the mean-square parameter estimation error may be obtained by including the higher order terms in the expansion. In many practical cases, however, the higher order terms are insignificant compared to the first-order terms. Note that the result is very similar to what was obtained for state estimation errors.

Because systematic instrument errors produce an additive term in the mean-square estimation error for θ, their effects can be analyzed one at a time. This then gives the sensitivities of parameter estimation errors to each component of instrument error. This is important when tradeoffs have to be established between different kinds of errors.

Parameter Estimation Errors When Systematic Errors Are Estimated Concurrently With Parameters

Since the likelihood function is $J(\theta,\phi)$, θ and ϕ estimates are obtained by simultaneous solution of

$$\frac{J(\hat{\theta},\hat{\phi})}{\partial\hat{\theta}}=0,\qquad \frac{\partial J(\hat{\theta},\hat{\phi})}{\partial\hat{\phi}}=0 \tag{31}$$

Again, by expanding Eq. (31) about the true values (θ,ϕ), it may be shown that the estimates (θ,ϕ) are unbiased to first order. The mean-square error of the estimate $\hat{\theta}$ can be shown to be

$$\mathrm{mse}(\tilde{\theta})=\underbrace{M_{11}^{-1}}_{\text{estimation error even when systematic errors are absent}}+\underbrace{M_{11}^{-1}M_{12}(M_{22}-M_{12}^T M_{11}^{-1}M_{12})^{-1}M_{12}^T M_{11}^{-1}}_{\text{additional estimation error because systematic errors also are estimated}} \tag{32}$$

The second term in the error equation is caused by the distribution of the information in the sensor measurements over a larger number of parameters, thus reducing the ac-

Table 3 Effect of systematic instrument errors on parameter estimation errors

	Bias	Variance	Mean-square error
Systematic errors set to a priori calibration values	$T_b(\phi-\phi_c)$ $T_b = M_{11}^{-1}M_{12}$	M_{11}^{-1}	$M_{11}^{-1}+T_b(\phi-\phi_c)\times(\phi-\phi_c)^T T_b^T$
Systematic errors identified simultaneously with state	0	$M_{11}^{-1}+T_b(M_{22}-M_{12}^T M_{11}^{-1}M_{12})^{-1}T_b^T$ $T_b = M_{11}^{-1}M_{12}$	$M_{11}^{-1}+T_b(M_{22}-M_{12}^T M_{11}^{-1}M_{12})^{-1}T_b^T$

curacy of each estimate. Note that the second term does not depend on the calibration values of systematic errors. This implies that a constant penalty is suffered when the error parameters are estimated, regardless of the error in the initial calibration. Table 3 summarizes the results.

Effect of Random Errors on Parameter Estimation Accuracy

The parameter estimation error covariance caused by random errors is M_{11}^{-1}. It is straightforward to show that[1]

$$M_{11}=\int_0^T \frac{\partial \nu^T(t)}{\partial \theta}R^{-1}\frac{\partial \nu(t)}{\partial \theta}\mathrm{d}t \tag{33}$$

If there are p instruments, and noise in any two instruments is uncorrelated, it is easy to see that

$$M_{11}=\sum_{i=1}^{p}\frac{1}{R_{ii}}\int_0^T\left(\frac{\partial \nu_i(t)}{\partial \theta}\right)^T\left(\frac{\partial \nu_i(t)}{\partial \theta}\right)\mathrm{d}t \tag{34}$$

where R_{ii} is the intensity of error in the ith instrument. It is clear from Eq. (34) that the contribution of each instrument toward the total information is additive. The gradients of innovations, in general, do depend upon measurement noise covariance, but the explicit effect is much more important. Equation (34) may be rewritten as

$$M_{11}=\sum_{i=1}^{p}\frac{\Delta M_{ii}}{R_{ii}} \tag{35}$$

Table 4 Orientations of gimbals with respect to vehicle axis system

Mechanical order	Freedom	Direction of axis
Innermost gimbal	Unlimited	Yaw
Second gimbal out	± 14.5 deg	Roll
Third gimbal out	Unlimited	Pitch
Outermost gimbal	Unlimited	Roll

Table 5 Description of platform error terms

	Term	Description
Component gyro	D_F	Specific force independent drift
	D_I, D_O, D_S	Drifts proportional to specific force
	D_{II}, D_{OO}, D_{SS} D_{IS}, D_{IO}, D_{OS}	Drift proportional to quadratic functions of specific force components
Accelerometer	K_o	Bias
	K_{1i}	Scale factor
	K_{2i}	Quadractic nonlinearity
	K_{3i}	Cubic nonlinearity
	β, γ	Orientation errors
	K_{ip}, K_{io}	Cross-product errors
	ν_A	Random noise
Resolver	K_ϕ	Scale factor error
	b_ϕ	Bias
	ν_ϕ	Random noise

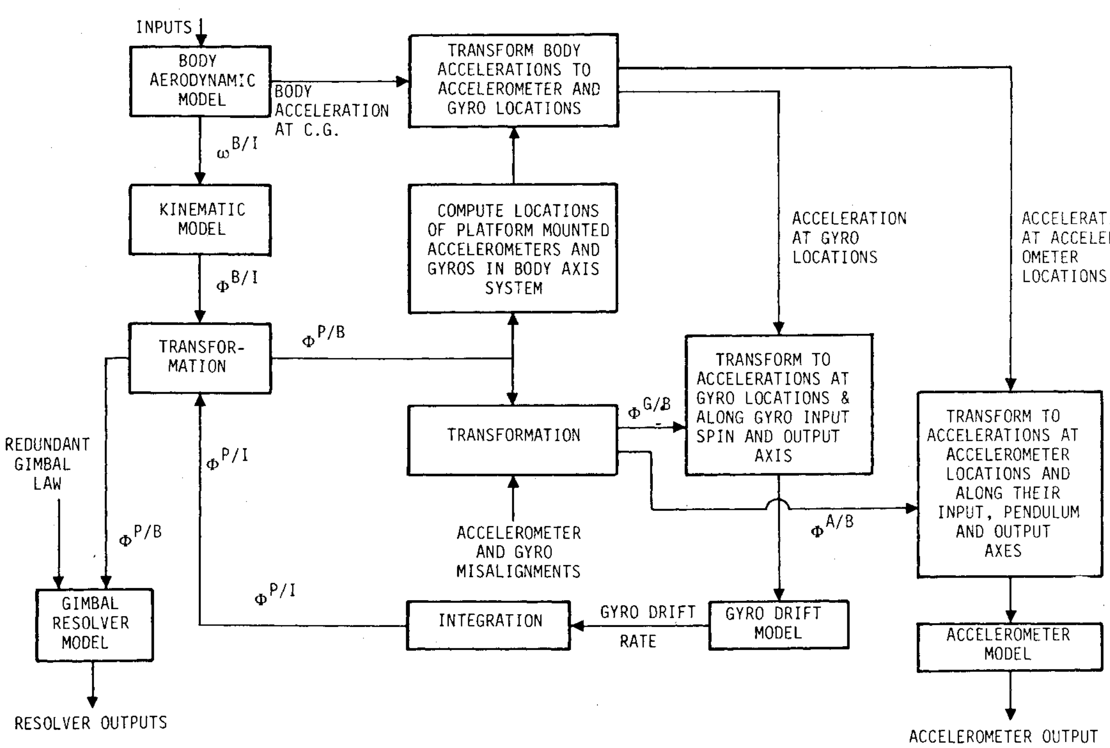

Fig. 1 Flow chart for evaluating effect of platform errors on aerodynamic coefficient estimation accuracy.

Note that ΔM_{ii} is nearly independent of R_{ii}. Therefore, the determination of the sensitivity of M_{II} to various components of instrument random errors is a straightforward procedure.

Example

This section presents an example to demonstrate the analysis procedure given previously. We show the effects of errors in a four-gimbal platform on estimates of nonlinear aerodynamic coefficients of a re-entry vehicle. Table 4 shows the relationships of the nominal gimbal orientations with respect to the vehicle axis system. The outputs are the four gimbal resolver angles and the three axis accelerometers. Figure 1 shows schematically the relationships between body motions and instrument outputs.

Instrument Error Models

The orientation gyros, accelerometers, and resolvers all have errors. The precession rate of gyro depends on specific forces along the three axes and is given approximately by (gyro precession causes platform precession)

$$\begin{aligned}\omega_I = {} & D_F + D_I(SF)_I + D_O(SF)_O + D_S(SF)_S \\ & + D_{II}(SF)_I^2 + D_{OO}(SF)_O^2 + D_{SS}(SF)_S^2 \\ & + D_{IS}(SF)_I(SF)_S + D_{IO}(SF)_O(SF)_I + D_{OS}(SF)_O(SF)_S \end{aligned} \tag{36}$$

where (SF) is specific force and I, O, and S refer to input, output, and spin axis. The accelerometer output contains bias and is affected by nonlinearities and misalignments. The indicated output is

$$\begin{aligned}A_{\text{ind}} = {} & K_o + K_{1i}A_i + K_{2i}A_i^2 + K_{3i}A_i^3 + \beta A_o \\ & - \gamma A_p + K_{ip}A_iA_p + K_{io}A_iA_o + v_A\end{aligned} \tag{37}$$

where i, p, and o refer to input, pendulum, and output axes, and β and γ are the misalignment angles. In addition, the resolver measurements contain scale factor, bias, and random errors. The resolver angle output is

$$\phi_m = (1 + k_\phi)\phi + b_\phi + v_\phi \tag{38}$$

where ϕ is the true resolver angle. Table 5 describes these error terms.

Aerodynamic Model

A nonlinear aerodynamic model for the longitudinal motion is analyzed. The normal force and pitch moment coefficients are

$$C_z = C_{z_o} + C_{z_\alpha}\alpha + C_{z_{\alpha^2}}\alpha^2 \tag{39a}$$

$$\begin{aligned}C_m = {} & C_{m_o} + C_{m_\alpha}\alpha + C_{m_\alpha}\delta + C_{m_{\alpha\delta}}\alpha\delta + C_{m_{\alpha^2}}\alpha^2 + C_{m_{\mu^3}}\mu^3 \\ & + C_{m_{\alpha^4}}\alpha^4 + C_{m_{\alpha\delta^2}}\alpha\delta^2 + C_{m_{\alpha^3}}\alpha^3 \\ & + C_{m_{\mu\alpha^2}}\mu\alpha^2 + C_{m_{\alpha^2\delta^3}}\alpha^2\delta^3 + C_{m_{\mu\alpha^3}}\mu\alpha^3\end{aligned} \tag{39b}$$

Here α is the angle of attack, δ is the control deflection, and μ is the Mach number. This model and the corresponding parameter values are based on the wind-tunnel data. It has been observed that the basic variations of C_m and C_z with α, δ, and μ in flight are similar to those in the wind tunnel, although specific model form and parameter values may be different. Therefore, this model forms a useful basis for instrument analysis and specifications.

Results

For the given flight path, the following results are computed: 1) effects of each systematic error term on each aerodynamic parameter (e.g., see Fig. 2); 2) effects of variations in random error (e.g., see Fig. 3); 3) contribution of each instrument toward total error in each aerodynamic parameter (e.g., see Fig. 4); and 4) total error in each parameter (e.g., see Table 6).

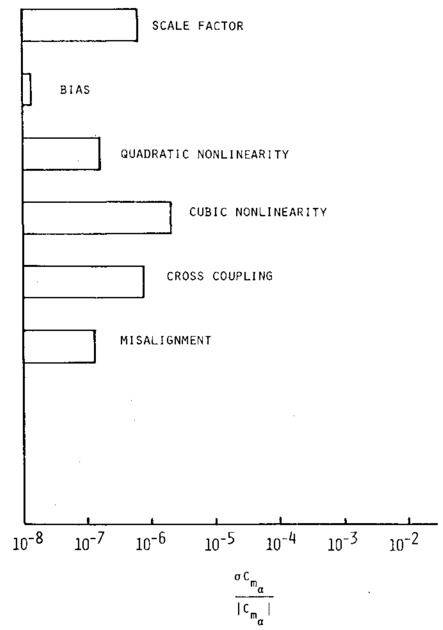

Fig. 2 Contributions of various vertical accelerometer error terms on C_{m_α} estimation error.

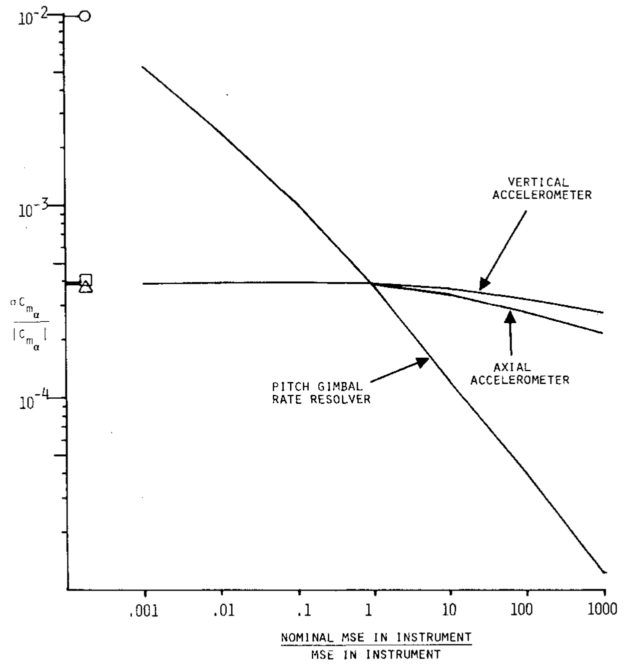

Fig. 3 Effect of random error in instruments on root mean square C_{m_α} estimation error.

Figure 2 shows that cubic nonlinearity and cross-coupling in accelerometer affect C_{m_α} estimation error more than other error sources. This helps comparison of different accelerometers with different combinations of individual errors. Figure 3 shows that it is more important to reduce random error in gimbal angles than in accelerometers. In fact, failure of gimbal error measurement could be very serious. Figure 4 indicates that, for nominal values of instrument errors, random errors are more important than systematic errors, and a hundredfold increase in systematic errors is not too serious.

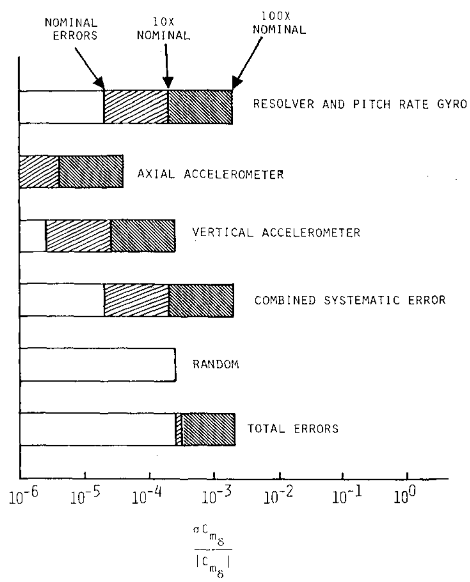

Fig. 4 Contribution of various instruments toward C_{m_α} estimation error.

Table 6 Summary of total parameter estimation errors[a]

Parameter	Percent error in parameter estimates: Systematic errors nominal	Systematic errors 10× nominal	Systematic errors 100× nominal
C_{m_0}	0.10	0.10	0.17
C_{m_δ}	0.027	0.033	0.20
C_{m_α}	0.041	0.061	0.45
$C_{m_{\alpha\delta}}$	0.11	0.12	0.17
$C_{m_{\alpha^2}}$	0.18	0.22	1.3
$C_{m_{\alpha^4}}$	0.097	0.11	0.53
$C_{m_{\alpha^2\mu}}$	0.28	0.30	1.2
$C_{m_{\alpha^2\delta^3}}$	0.062	0.063	0.13
$C_{m_{\alpha^3\mu}}$	0.71	0.94	6.3
C_{z_0}	0.53	0.60	2.9
C_{z_α}	0.70	0.74	2.5
$C_{z_{\alpha^2}}$	4.4	6.5	6.8

[a] Random errors nominal.

(A similar increase in random errors would be very undesirable.) Overall errors under design and off-design conditions are given in Table 6.

The errors in parameter estimates because of platform errors are quite small. To get total error estimates, we must add to values given in Table 6 effects of 1) error in knowledge of atmospheric density; 2) errors in estimates of mass, moments of inertia, and principal axes; 3) instrument location errors; 4) finite sampling rate and word length; and 5) errors in measurement of input signals. These error sources were analyzed in exactly the same manner as the platform errors using the technique of this paper, although the details of these analyses have been left out of the paper for sake of brevity. These sources add 2 to 15% more to parameter estimation errors. The point of this entire exercise is to show that the platform is a highly accurate device and does not contribute substantially to total estimation error in parameters. As far as the parameter estimation function is concerned, it is more important to increase word length and to get better measurements of control deflection and weather data than it is to use better and more expensive platforms.

Conclusions

This paper discussed analytical techniques for the determination of the influence of instrument random and systematic errors on parameter and state estimation accuracies. These techniques are based on the information matrix formulation, which leads to significantly simpler algorithms. A major hindrance to the development of such approaches has been the traditional use of covariance matrix in most estimation problems. The algorithms resulting from information matrix analysis also provide good insight into the nature and cause of errors and provide rapid means of establishing tradeoffs between computation requirements and estimation accuracy.

The determination of errors in state and parameter estimates requires a "true" model and an "estimation" model. The true model is more accurate than the estimation model to the extent that it recognizes possibilities of errors in the estimation model. As mentioned previously, these errors might arise in the state equations (e.g., density variations, actuator lags, wind speed) or in the measurement equations (e.g., biases, nonlinearities). Otherwise, the true model and the estimation model are equivalent. Therefore, the requirement of the true model availability is not considered a handicap in instrumentation analysis for engineering problems.

The basic advantage of the analytical technique of this paper is that it takes much less computer time than conventional Monte Carlo type methods. (The reduction for complex systems could be several orders of magnitude.) Application of these methods should enable a thorough evaluation of the sensor/channel characteristics prior to system design and prototype test. Our experience indicates that this evaluation, at an early stage, often reduces system cost by simplifying the sensor subsystems and could enhance state/parameter estimation accuracies. Critical "bugs" or instrument mismatches also are eliminated. Custom sensor subsystem sometimes may be specified to meet the requirements of a particular mission better.

Acknowledgments

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